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Director orientation in nematic liquid crystals using crossed electric and magnetic fields

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Abstract. A theoretical investigation is made of static and dynamic effects when a nematic liquid crystal is subjected to crossed electric and magnetic fields. In the static problem a twist-wall solution is discussed for a semi-infinite sample of nematic; a control parameter, q, describes the relationship between the fields and their crossed angle and is used to characterize the solution. For an infinite sample of nematic this parameter also turns out to characterize the types of solution (travelling waves) which are available for the nonlinear dynamic equation when certain approximations are made. The type of solution which occurs is shown to depend crucially on the boundary conditions, the relative magnitudes of the electric and magnetic fields and their crossed angle.

1. Introduction

Liquid crystals usually consist of elongated molecules where the long molecular axes locally adopt one common direction in space. This direction is described by the unit vector $\boldsymbol{n}$, commonly called the director. The orientation of the director in liquid crystals can be driven by either an electric field $\boldsymbol{E}$ or magnetic field $\boldsymbol{H}$ or a crossed combination of both. Motivated by de Gennes [1] and the review article by Lam [2], we study a layer of nematic liquid crystal in the xy-plane which is assumed to have a twist wall in the z-direction when the crossed fields are in the xy-plane. We investigate two related problems. The first problem examines static solutions in a semi-infinite sample where the layer is assumed to have the director strongly anchored parallel to the x-axis (that is, on the lower surface plate at $z = 0$ in the direction of $\boldsymbol{H}$) and bounded above by a free surface, considered here to be at $z = +\infty$. The second problem looks at time-dependent solutions in an infinite sample. It turns out that the critical parameters derived in the static problem are also useful when looking for travelling-wave solutions in the time-dependent problem.

Specifically, we set

$$\boldsymbol{n} = (\cos\phi(z,t), \sin\phi(z,t), 0) \tag{1.1}$$

where ϕ is the angle $\boldsymbol{n}$ makes with the x-axis and introduce the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields as

$$\boldsymbol{E} = E(\cos\beta, \sin\beta, 0) \tag{1.2}$$

$$\boldsymbol{H} = H(1, 0, 0) \tag{1.3}$$

where E and H are the magnitudes of the fields and β is the angle between them with $0 \leqslant \beta \leqslant \pi/2$.

For the ansatz given in (1.1) the bulk energy is [3]

$$w = \tfrac{1}{2}K_2(\boldsymbol{n}\cdot\nabla\times\boldsymbol{n})^2 \tag{1.4}$$

where $K_2 > 0$ is the twist elastic constant. The corresponding combined electric and magnetic energy is

$$\psi = -\tfrac{1}{2}\chi_a(\boldsymbol{n}\cdot\boldsymbol{H})^2 - \tfrac{1}{2}\epsilon_a\epsilon_0(\boldsymbol{n}\cdot\boldsymbol{E})^2 \tag{1.5}$$

where orientation-independent terms have been excluded. The diamagnetic anisotropy χ_a and the dielectric anisotropy ϵ_a are assumed to be positive; ϵ_0 is the permittivity of free space. In the absence of bulk flow the relevant dynamic equations for the director motion in the usual cartesian component form for $i = 1, 2, 3$ are

$$\left(\frac{\partial w}{\partial n_{i,j}}\right)_{,j} - \frac{\partial w}{\partial n_i} - \frac{\partial\psi}{\partial n_i} - \gamma_1\frac{\partial n_i}{\partial t} + \gamma n_i = 0 \tag{1.6}$$

where γ_1 is the twist viscosity coefficient and γ is a scalar Lagarange multiplier [4, p 238] (note that our form for ψ is opposite in sign to that used in [4]). We can eliminate the scalar γ by multiplying $(1.6)_1$ by $\sin\phi$ and $(1.6)_2$ by $\cos\phi$ and subtracting the resulting equations to yield the dynamic equation

$$\gamma_1\phi_t = K_2\phi_{zz} - \tfrac{1}{2}\chi_a H^2\sin(2\phi) - \tfrac{1}{2}\epsilon_a\epsilon_0 E^2\sin 2(\phi-\beta). \tag{1.7}$$

This equation is of the form mentioned by Lam [2, p 32] when $H = 0$. Faetti *et al* [5] have also examined (1.7) when the elastic term is absent and the crossed fields are switched on and off alternately. In this article we intend to include both fields and the elastic term.

There are no known exact solutions to the nonlinear equation (1.7). Nevertheless, the qualitative nature of the solutions can be obtained by making some approximations. In section 2 we briefly discuss the full nonlinear static twist solution to (1.7) in the semi-infinite sample with the boundary condition $\phi \equiv 0$ at $z = 0$. In section 3 we approximate the right-hand side of (1.7) up to cubic order in ϕ to search for time-dependent soliton-like travelling-wave solutions for ϕ close to $\pi/2$, making use of the critical parameters introduced in the static problem. Through these approaches we can establish some insight into the behaviour of solutions to (1.7) and perceive the possible influences of $\boldsymbol{E}$, $\boldsymbol{H}$ and β on the types of soliton-like orientation of $\boldsymbol{n}$.

2. Static solutions

For the static case, equation (1.7) can be written as

$$\xi^2\phi_{zz} = \tfrac{1}{2}\sin(2\phi - q) \tag{2.1}$$

where

$$\xi = \sqrt{K_2}(\epsilon_a^2\epsilon_0^2E^4 + \chi_a^2H^4 + 2\epsilon_a\epsilon_0\chi_a E^2H^2\cos(2\beta))^{-1/4} \tag{2.2}$$

$$q = \tan^{-1}\left(\frac{\epsilon_a\epsilon_0E^2\sin(2\beta)}{\chi_aH^2 + \epsilon_a\epsilon_0E^2\cos(2\beta)}\right). \tag{2.3}$$

It will be seen that the parameter q controls the type of solutions we find; this is to be expected since q gives the key information on the relationship between the control parameters E, H and β. It is of course possible to consider equation (1.5) as a single-field term, but this would only include a special case of a particular fixed angle between the fields and then equation (1.7) (rewritten as equation (2.1)) would appear in a simpler form. The principal reason for using two separate yet combined fields rather than one single field is that experimentally the electric and magnetic fields have a measurable physical angle between them, as discussed and pictured in [5] where the alternate presence of these fields is considered. This means that the above description of such problems is more general since the influence of either field can be assessed via the single control parameter q in equation (2.3). A single field would initially permit a simpler description, but the resulting analysis from section 3 onwards would essentially remain unchanged, while the interplay of the competing fields would remain hidden. The above approach simplifies the combined crossed-field problem and permits a fuller discussion on the effects of the relative magnitudes of the fields upon the types of available solutions.

Equation (2.1) is a version of the static sine–Gordon equation which has the exact twist solution

$$\phi(z) = q/2 - \pi + 2\tan^{-1}(\exp[-(z - z_0)/\xi]). \tag{2.4}$$

Imposing the boundary condition $\phi \equiv 0$ at $z = 0$ forces the centre of the twist to be at

$$z_0 = \xi \ln[\tan(\pi/2 - q/4)]. \tag{2.5}$$

Clearly z_0 increases as q decreases to zero and ϕ decreases from zero to $q/2 - \pi$ (that is, the director twists through a total angle of $\pi - q/2$). From (2.3) the physical interpretation is that the twist will unwind towards the free surface as either the magnitude of the $\boldsymbol{E}$ field is decreased (while $H > 0$ is fixed) or, in general, the angle β between the fields tends to zero. There are two cases to consider in detail:

Case (i) $\chi_a H^2 > \epsilon_a \epsilon_0 E^2$. Here, from (2.3), we must have $0 \leqslant q < \pi/2$ with the maximum value of q occurring at the critical angle β_1 given by

$$\beta_1 = \frac{1}{2}\cos^{-1}\left(\frac{-\epsilon_a\epsilon_0 E^2}{\chi_a H^2}\right) \tag{2.6}$$

leading to

$$q_{\max} = \tan^{-1}\left(\epsilon_a\epsilon_0 E^2(\chi_a^2 H^4 - \epsilon_a^2\epsilon_0^2 E^4)^{-1/2}\right). \tag{2.7}$$

Therefore as β decreases from β_1 to zero or increases from β_1 to $\pi/2$ the centre of twist z_0 increases smoothly to infinity since q tends to zero. The minimum value of z_0 is at $\beta = \beta_1$ and is given by equation (2.5) when $q = q_{\max}$.

Case (ii) $\chi_a H^2 \leqslant \epsilon_a \epsilon_0 E^2$. In this case, from (2.3), there is a critical angle β_2 given by

$$\beta_2 = \frac{1}{2}\cos^{-1}\left(\frac{-\chi_a H^2}{\epsilon_a\epsilon_0 E^2}\right) \tag{2.8}$$

where $q = \pi/2$. It is clear that q is positive for $0 \leqslant \beta \leqslant \beta_2$ and therefore equation (2.5) gives positive values for z_0. For $\beta > \beta_2$ equation (2.3) shows q to be negative which then forces (2.5) to have no solutions for z_0. Therefore, provided $\beta \leqslant \beta_2$ the centre of twist will increase to infinity as β decreases to zero. The minimum value for z_0 will be at $\beta = \beta_2$ where

$$z_0 = \xi \ln\left[\tan(3\pi/8)\right]. \tag{2.9}$$

3. Travelling-wave solutions

Employing the notation of section 2, equation (1.7) can be written as

$$\eta\phi_t = \xi^2\phi_{zz} - \tfrac{1}{2}\sin(2\phi - q) \tag{3.1}$$

where

$$\eta = \gamma_1(\epsilon_a^2\epsilon_0^2 E^4 + \chi_a^2 H^4 + 2\epsilon_a\epsilon_0\chi_a E^2 H^2\cos(2\beta))^{-1/2}. \tag{3.2}$$

We now seek solutions for ϕ close to $\pi/2$ and set

$$\hat{\phi} = \phi - \frac{\pi}{2} \tag{3.3}$$

which effectively changes the first negative sign in (3.1) to a plus sign. Equation (3.1) can then be approximated by Taylor expanding the sine term up to cubic order in $\hat{\phi}$, making no restrictions on the control parameter q. This results in

$$\eta\hat{\phi}_t = \xi^2\hat{\phi}_{zz} - \tfrac{2}{3}(\cos q)F(\hat{\phi}, q) \tag{3.4}$$

where

$$F(\hat{\phi}, q) = \hat{\phi}^3 - \tfrac{3}{2}(\tan q)\hat{\phi}^2 - \tfrac{3}{2}\hat{\phi} + \tfrac{3}{4}\tan q. \tag{3.5}$$

Rescaling with

$$T = t\frac{2\cos q}{3\eta} \tag{3.6}$$

$$Z = z\left(\frac{2\cos q}{3\xi^2}\right)^{1/2} \tag{3.7}$$

then leads to

$$\hat{\phi}_T = \hat{\phi}_{ZZ} - F(\hat{\phi}, q). \tag{3.8}$$

Of course, these transformations will only be valid for $q \neq \pm\pi/2$ which, by equation (2.3), is always true if $\chi_a H^2 > \epsilon_a\epsilon_0 E^2$ or $\chi_a H^2 \neq -\epsilon_a\epsilon_0 E^2\cos(2\beta)$; when $q = \pm\pi/2$ the right-hand side of equation (3.4) becomes a quadratic in $\hat{\phi}$. We shall first concentrate on the case when $|q| < \pi/2$ and discuss the quadratic case later below.

3.1. The cubic approximation

As discussed by Lam [2, p 20], finding the roots in $\hat{\phi}$ of F may allow us to construct exact travelling-wave solutions for the solution $\hat{\phi}$ of (3.8) whenever the roots are real. It is simple to check that the cubic discriminant for F is [6, p 17]

$$\Delta F = -\tfrac{1}{8} - \tfrac{3}{16}\tan^2 q - \tfrac{3}{32}\tan^4 q \tag{3.9}$$

which is clearly strictly negative for all $|q| < \pi/2$ and hence $F(\hat{\phi}, q)$ has exactly three distinct real roots. Using the method of Cardano and Lagrange [7] these roots are found to be

$$\phi_1 = (\tan^2 q + 2)^{1/2}\cos(\theta_0) + \tfrac{1}{2}\tan q \tag{3.10}$$

$$\phi_2 = (\tan^2 q + 2)^{1/2}\cos(\theta_0 + 4\pi/3) + \tfrac{1}{2}\tan q \tag{3.11}$$

$$\phi_3 = (\tan^2 q + 2)^{1/2}\cos(\theta_0 + 2\pi/3) + \tfrac{1}{2}\tan q \tag{3.12}$$

where

$$\theta_0 = \frac{1}{3}\tan^{-1}\left(\frac{(6\tan^4 q + 12\tan^2 q + 8)^{1/2}}{\tan^3 q}\right). \tag{3.13}$$

In all possible cases, $-\pi/2 < q < \pi/2$ and hence $-\pi/6 \leqslant \theta_0 < 0$ or $0 < \theta_0 \leqslant \pi/6$ with $\theta_0 = -\pi/6$ when q approaches zero from the left and $\theta_0 = \pi/6$ when q approaches zero from the right ($\theta_0 = 0$ only when $q = \pm\pi/2$). Straightforward consideration of the cosine function leads to the following inequalities with their corresponding conditions on the control parameter q:

$$\phi_1 > 0 \qquad (|q| < \pi/2) \tag{3.14}$$

$$\phi_3 < \phi_2 < \phi_1 \qquad (0^+ < q < \pi/2) \tag{3.15}$$

$$\phi_2 < \phi_3 < \phi_1 \qquad (-\pi/2 < q < 0^-) \tag{3.16}$$

$$\phi_2, \phi_3 < \phi_1 \qquad (q = 0). \tag{3.17}$$

We shall restrict our attention to the case in (3.15) when $0 \leqslant q < \pi/2$, the case for negative q being analogous with the rôles of ϕ_2 and ϕ_3 interchanged (it will be shown below that $\hat{\phi}$ converges to the same solution as q tends to 0^+ or 0^-).

To find travelling-wave solutions we introduce the variable τ defined by

$$\tau = Z - cT + Z_0 \tag{3.18}$$

where Z_0 is an arbitrary constant and c is a constant to be determined. Equation (3.8) then becomes

$$\hat{\phi}_{\tau\tau} + c\hat{\phi}_\tau = F(\hat{\phi}, q). \tag{3.19}$$

We are particularly interested in solutions where $c > 0$. There are three types of solution to consider, similar to those defined by Lam. The type of solution which is physically relevant is the one which is stable to small perturbations and whose boundary conditions at $z = \pm\infty$ are closest to the boundary conditions being modelled in a given particular problem. These soliton-like solutions discussed here require third order expansions in $\hat{\phi}$ for the different solution types A, B and C. These are characterized by the single control parameter q and the cubic roots of F, given by (3.10)–(3.12). Although similar travelling waves are discussed by Lam, here the solutions are interesting in that they are directly seen to be related to changes in the angle β via the control parameter q. Despite exhibiting different forms of travelling waves, the physically prevalent solutions are not identified in this present paper; a discusion of the stability of solutions is beyond the scope of this present work where the techniques and consequences of finding solutions will be of wider interest. Work in progress [10] addresses this problem and shows that in special cases of combined fields only one of the three main types of solution (exactly which one depends upon the choice of various physical parameters) is generally stable to small perturbations, this solution being identified with the physical solution.

3.1.1. Type A travelling waves. We first note that ϕ_1 is always greater than ϕ_2 or ϕ_3. For $q > 0$ type A solutions occur when $\hat{\phi}$ travels from ϕ_1 to ϕ_2 as $\tau \to \infty$. The solution to (3.19) with this behaviour is

$$\hat{\phi} = (\phi_1 - \phi_2)\left\{1 + \exp\left[\frac{1}{\sqrt{2}}(\phi_1 - \phi_2)\tau\right]\right\}^{-1} + \phi_2 \tag{3.20}$$

where, by (3.15), we have

$$c = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2 - 2\phi_3) > 0. \tag{3.21}$$

Direct substitution of (3.20) into (3.19) verifies this to be the solution. This solution can be expressed in the original variables via (3.3), (3.6), (3.7), (3.18) as

$$\phi = (\phi_1 - \phi_2)\{1 + \exp[a(z - vt + z_0)]\}^{-1} + \phi_2 + \pi/2 \tag{3.22}$$

where z_0 is an arbitrary constant and

$$a = (\phi_1 - \phi_2)\frac{1}{\xi}\left(\frac{\cos q}{3}\right)^{1/2} \tag{3.23}$$

$$v = (\phi_1 + \phi_2 - 2\phi_3)\frac{\xi}{\eta}\left(\frac{\cos q}{3}\right)^{1/2}. \tag{3.24}$$

This solution is valid when ϕ is close to $\pi/2$.

Following the definition in [2, p 42], the wave width W_A of such solutions is the modulus of the inverse of the coefficient of z and therefore

$$\begin{aligned} W_A(q) &= \sqrt{3}\xi(\phi_1 - \phi_2)^{-1}(\cos q)^{-1/2} \\ &= \xi(\tan^2 q + 2)^{-1/2}(\cos q)^{-1/2}[\sin(\theta_0 + 2\pi/3)]^{-1}. \end{aligned} \tag{3.25}$$

$W_A(q)$ shows that large-amplitude travelling-wave solutions are narrower and travel faster than the small-amplitude solutions. This definition of wave width is similar to considering the distance (in τ) over which the solution (3.22) changes by $\frac{1}{2}|\phi(+\infty) - \phi(-\infty)|$ across the 'centre' of the wave at $\tau = 0$. This distance is easily computed as $2\ln(3)W_A(q)$, which is clearly proportional to $W_A(q)$. Similar widths have been discussed by Helfrich [8] for nematics and Stewart and Raj [9] in the case of smectics.

Solutions for negative q are as in equations (3.20) to (3.25) except that the roles of ϕ_2 and ϕ_3 are interchanged and $2\pi/3$ is replaced by $\pi/3$ in equation (3.25). From (3.16) it is also true in this case that $c > 0$.

As q tends to 0^+ we see that

$$\phi_1 \to \sqrt{3/2} \qquad \phi_2 \to 0 \qquad \phi_3 \to -\sqrt{3/2} \tag{3.26}$$

and as q tends to 0^-

$$\phi_1 \to \sqrt{3/2} \qquad \phi_2 \to -\sqrt{3/2} \qquad \phi_3 \to 0. \tag{3.27}$$

It is then seen that $c \to 3\sqrt{3}/2$ as $q \to 0$ and therefore the solution ϕ converges to

$$\phi = \sqrt{3/2}\left\{1 + \exp\left[\frac{1}{\sqrt{2}\xi}\left(z - \frac{3\xi}{\sqrt{2}\eta}t + z_1\right)\right]\right\}^{-1} + \pi/2 \tag{3.28}$$

(z_1 an arbitrary constant) as $q \to 0$ with $W_A(0) = \sqrt{2}\xi$, this being the maximum wave width.

3.1.2. Type B travelling waves. For $q > 0$ type B travelling waves occur when $\hat{\phi}$ travels from ϕ_3 to ϕ_2 as $\tau \to \infty$. The solution to (3.19) is then

$$\hat{\phi} = (\phi_2 - \phi_3)\left\{1 + \exp\left[-\frac{1}{\sqrt{2}}(\phi_2 - \phi_3)\tau\right]\right\}^{-1} + \phi_3 \tag{3.29}$$

where, by (3.15), $(\phi_2 - \phi_3) > 0$ and

$$c = -\frac{1}{\sqrt{2}}(\phi_2 + \phi_3 - 2\phi_1) > 0. \tag{3.30}$$

As above, the solution can be expressed in the original variables as

$$\phi = (\phi_2 - \phi_3)\{1 + \exp[-a(z - vt + z_0)]\}^{-1} + \phi_3 + \pi/2 \tag{3.31}$$

with z_0 an arbitrary constant and

$$a = (\phi_2 - \phi_3)\frac{1}{\xi}\left(\frac{\cos q}{3}\right)^{1/2} \tag{3.32}$$

$$v = -(\phi_2 + \phi_3 - 2\phi_1)\frac{\xi}{\eta}\left(\frac{\cos q}{3}\right)^{1/2}. \tag{3.33}$$

The wave width in this case is

$$\begin{aligned} W_B(q) &= \sqrt{3}\xi(\phi_2 - \phi_3)^{-1}(\cos q)^{-1/2} \\ &= \xi(\tan^2 q + 2)^{-1/2}(\cos q)^{-1/2}[\sin(\theta_0)]^{-1}. \end{aligned} \tag{3.34}$$

For $q < 0$ we interchange ϕ_2 and ϕ_3 in all of the above and replace θ_0 by $-\theta_0$ in equation (3.34).

By (3.26) and (3.27), $c \to 3\sqrt{3/2}$ as $q \to 0$ and the solution ϕ converges to

$$\phi = \sqrt{3/2}\left\{1 + \exp\left[-\frac{1}{\sqrt{2}\xi}\left(z - \frac{3\xi}{\sqrt{2}\eta}t + z_1\right)\right]\right\}^{-1} - \sqrt{3/2} + \pi/2 \tag{3.35}$$

where z_1 is an arbitrary constant. Here, $W_B(0) = \sqrt{2}\xi$, this being the minimum wave width, by equation (3.34) (the wavewidth grows to infinity as $q \to \pm\pi/2$, that is, as $\theta_0 \to 0$).

3.1.3. Type C travelling waves. Type C travelling waves occur when $\hat{\phi}$ travels from ϕ_1 to ϕ_3 as $\tau \to \infty$ for $q > 0$. The solution to (3.19) is then

$$\hat{\phi} = (\phi_1 - \phi_3)\left\{1 + \exp\left[\frac{1}{\sqrt{2}}(\phi_1 - \phi_3)\tau\right]\right\}^{-1} + \phi_3 \tag{3.36}$$

where, by (3.15), $(\phi_1 - \phi_3) > 0$ and

$$c = \frac{1}{\sqrt{2}}(\phi_1 + \phi_3 - 2\phi_2) > 0. \tag{3.37}$$

This gives the solution in the original variables as

$$\phi = (\phi_1 - \phi_3)\{1 + \exp[a(z - vt + z_0)]\}^{-1} + \phi_3 + \pi/2 \tag{3.38}$$

with z_0 an arbitrary constant and

$$a = (\phi_1 - \phi_3)\frac{1}{\xi}\left(\frac{\cos q}{3}\right)^{1/2} \tag{3.39}$$

$$v = (\phi_1 + \phi_3 - 2\phi_2)\frac{\xi}{\eta}\left(\frac{\cos q}{3}\right)^{1/2}. \tag{3.40}$$

The wave width is

$$\begin{aligned} W_C(q) &= \sqrt{3}\xi(\phi_1 - \phi_3)^{-1}(\cos q)^{-1/2} \\ &= \xi(\tan^2 q + 2)^{-1/2}(\cos q)^{-1/2}[\sin(\theta_0 + \pi/3)]^{-1}. \end{aligned} \tag{3.41}$$

If $q < 0$ then we replace ϕ_3 by ϕ_2 in equations (3.36) to (3.41) and replace $\pi/3$ by $2\pi/3$ in equation (3.41).

As $q \to 0$ the wave speed $c \to 0$, by (3.26) and (3.27). The travelling wave becomes a static domain wall, namely,

$$\phi = \sqrt{6}\left\{1+\exp[\sqrt{2}z/\xi + z_1)]\right\}^{-1} - \sqrt{3/2} + \pi/2. \tag{3.42}$$

z_1 an arbitrary constant, whose width equals the maximum wave width $W_C(0) = \xi/\sqrt{2}$. This is in contrast to type A and B travelling waves which never become static for any $|q| < \pi/2$.

3.2. *The quadratic approximation*

As was mentioned above, when $q = \pm\pi/2$ (that is, when $\chi_a H^2$ approaches $-\epsilon_a\epsilon_0 E^2\cos(2\beta)$) the right-hand side of equation (3.4) becomes (to cubic order) a quadratic in $\hat{\phi}$. When $q = \pi/2$ equation (3.4) is

$$\eta\hat{\phi}_t = \xi^2\hat{\phi}_{zz} + \hat{\phi}^2 - \tfrac{1}{2} \tag{3.43}$$

which has the exact travelling-wave solution

$$\hat{\phi} = 1/\sqrt{2} - \sqrt{2}\left\{1+\exp\left[\frac{2^{1/4}}{\sqrt{6}\xi}\left(z - \frac{5\xi 2^{1/4}}{\sqrt{6}\eta}t + z_0\right)\right]\right\}^{-2}. \tag{3.44}$$

This leads to an approximate solution (near $\pi/2$) of (3.1)

$$\phi = 1/\sqrt{2} + \pi/2 - \sqrt{2}\left\{1+\exp\left[\frac{2^{1/4}}{\sqrt{6}\xi}\left(z - \frac{5\xi 2^{1/4}}{\sqrt{6}\eta}t + z_0\right)\right]\right\}^{-2} \tag{3.45}$$

where the wave width is $W = \xi 2^{-1/4}\sqrt{6}$. When $q = -\pi/2$ equation (3.4) is

$$\eta\hat{\phi}_t = \xi^2\hat{\phi}_{zz} - \hat{\phi}^2 + \tfrac{1}{2} \tag{3.46}$$

which ultimately yields the corresponding approximate solution for ϕ as

$$\phi = -1/\sqrt{2} + \pi/2 + \sqrt{2}\left\{1+\exp\left[\frac{2^{1/4}}{\sqrt{6}\xi}\left(z - \frac{5\xi 2^{1/4}}{\sqrt{6}\eta}t + z_0\right)\right]\right\}^{-2} \tag{3.47}$$

which has the same wave width as (3.45). This shows that as the angle β approaches β_2^- from below (defined at equation (2.8)), $q > 0$ and the solution for ϕ sufficiently close to $\pi/2$ is therefore given by (3.45); this solution travels (in the variable τ) from $\pi/2 - 1/\sqrt{2}$ to $\pi/2 + 1/\sqrt{2}$. On the other hand, when β approaches β_2^+ from above the solution travels from $\pi/2 + 1/\sqrt{2}$ to $\pi/2 - 1/\sqrt{2}$; this follows from solution (3.47) when q is negative. One interpretation of this is that when $\chi_a H^2 \leqslant \epsilon_a\epsilon_a E^2$ then β_2 is the critical angle where, as the angle β increases across β_2, the solution for ϕ close to $\pi/2$ flips from an anti-kink to a kink solution. This type of behaviour is not possible if $\chi_a H^2 > \epsilon_a\epsilon_a E^2$ where $q \neq \pm\pi/2$ and the earlier cubic approximations hold.

It should also be noted that if we approximate ϕ to quadratic order close to zero (rather than close to $\pi/2$) then, for $q > 0$, equation (3.1) can be written as

$$\eta\phi_t = \xi^2\phi_{zz} - (\phi - \phi_1)(\phi - \phi_2)\sin q \tag{3.48}$$

where

$$\phi_1 = -\tfrac{1}{2}\cot q + \tfrac{1}{2}\sqrt{\cot^2 q + 2} \tag{3.49}$$

$$\phi_2 = -\tfrac{1}{2}\cot q - \tfrac{1}{2}\sqrt{\cot^2 q + 2}. \tag{3.50}$$

The resulting solution is easily verified as being

$$\phi = (\phi_1 - \phi_2)\{1 + \exp[a(z - vt + z_0)]\}^{-2} + \phi_2 \tag{3.51}$$

where z_0 is arbitrary and

$$a = (\phi_1 - \phi_2)^{1/2}\xi^{-1}\left(\frac{\sin q}{6}\right)^{1/2} \tag{3.52}$$

$$v = (\phi_1 - \phi_2)^{1/2}\frac{5\xi}{\eta}\left(\frac{\sin q}{6}\right)^{1/2}. \tag{3.53}$$

The wall width in this case is simply $W(q) = \xi\sqrt{6}(1 + \sin^2 q)^{-1/4}$.

4. Discussion

We have shown that there are static twist-wall phenomena and travelling waves when a nematic liquid crystal is subjected to crossed electric and magnetic fields. Static solutions were derived for a semi-infinite sample and various travelling waves were shown to occur in an infinite sample for solutions approximating the nonlinear dynamic equation (3.1).

In general, for $\chi_a H^2 > \epsilon_a\epsilon_0 E^2$, it was shown that the centre of twist in the static solution moves out towards infinity (that is, the twist in the sample effectively unwinds) as the angle β between the two fields approaches zero or $\pi/2$; the minimum value for the centre of twist occurs at the critical angle β_1 in equation (2.6) and is given by z_0 via equations (2.5) and (2.7). From an experimental point of view, the sample could have a fixed electric field applied across it and then be rotated in the presence of a magnetic field. The distance of the centre of twist from the bounding plate, z_0, would be observed to be a minimum at the critical angle β_1 given in equation (2.6), thereby establishing relationships between the physical constants K_2, χ_a and $\epsilon_a\epsilon_0$ by means of equations (2.2), (2.5), (2.6) and (2.7). Similarly, if $\chi_a H^2 \leqslant \epsilon_a\epsilon_0 E^2$ then the centre of twist moves out to infinity as β approaches zero. The minimum value for z_0 in this case occurs at the angle β_2 given at equation (2.8) where z_0 is given by (2.9). The control parameter q defined in equation (2.3) conveniently describes the link between the strengths of the two fields and the angle between them. A comparison of the relative magnitudes of electric and magnetic fields used for observing the reorientation of the director has been made experimentally and discussed by Carr [11].

In the infinite sample three general types of travelling waves were found when equation (3.1) was approximated to cubic order in ϕ near $\phi = \pi/2$. These solutions are completely characterized by the parameter q when $|q| < \pi/2$. One of these travelling waves, type C, was shown to reduce to a static solution as $q \to 0$. For the special case where $q = \pm\pi/2$ it was shown that the cubic approximation to (3.1) became a quadratic and that if $\chi_a H^2 \leqslant \epsilon_a\epsilon_0 E^2$ then the solution could switch from an anti-kink to a kink solution as the angle between the fields crossed the critical angle β_2. In making the approximations introduced in section 3 care should be taken in the physical relevance of the q parameter since the magnitude of q controls the magnitude of the roots ϕ_1, ϕ_2, ϕ_3 and hence the amplitudes of the derived travelling waves. Matching these roots to approximate the physical boundary conditions should be of benefit when modelling an experimental set-up. When ϕ is close to zero the solution (3.51) was found for the quadratic approximation to (3.1). For ϕ near zero the cubic approximations do not lead to solutions similar to those for ϕ close to $\pi/2$: the change in sign of the coefficient of ϕ^3 invalidates the methods employed above in the construction of travelling-wave solutions.

In deriving the dynamic equation (1.7) the general effects of bulk flow have been ignored, this being the usual procedure when the dynamics of the director n are first investigated. The inclusion of fluid flow is a natural next step in the crossed fields problem. In this respect it should be mentioned that Carr and McClymer [12] have experimentally observed transverse fluid flow in a nematic liquid crystal subjected to crossed electric and magnetic fields; the removal of the electric field was seen to produce defects in the nematic alignment. The theoretical modelling of these defects would be of great interest.

References

[1] de Gennes P G 1971 *J. Physique* **32** 789

[2] Lam L 1992 *Solitons in Liquid Crystals* ed L Lam and J Prost (Berlin: Springer) pp 9–50

[3] de Gennes P G 1974 *The Physics of Liquid Crystals* (Oxford: Clarendon)

[4] Leslie F M 1987 *Theory and Applications of Liquid Crystals* ed J L Ericksen and D Kinderlehrer (Berlin: Springer)

[5] Faetti S, Gammaitoni L, Martinelli M and Rolla P A 1991 *Liq. Cryst.* **9** 133

[6] Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover)

[7] Allenby R B J T 1983 *Rings, Fields and Groups* (London: Edward)

[8] Helfrich W 1968 *Phys. Rev. Lett.* **21** 1518

[9] Stewart I W and Raj N 1990 *Mol. Cryst. Liq. Cryst.* **185** 47

[10] Stewart I W and T R Faulkner 1995 On the stability of twist walls induced by crossed electric and magnetic fields in nematic liquid crystals, to appear

[11] Carr E F 1969 *Mol. Cryst. Liq. Cryst.* **7** 253

[12] Carr E F and McClymer J P 1990 *Mol. Cryst. Liq. Cryst.* **182B** 245